Multistage Global Search Using Various Scalarization Schemes in Multicriterial Optimization Problems

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In this paper, an approach, in which the decision making problems are reduced to solving the multicriterial time-consuming global optimization problems is proposed. The developed approach includes various methods of scalarization of the vector criteria, the dimensionality reduction with the use of the Peano space-filling curves, and the efficient global search algorithms. In the course of computations, the optimization problem statements and the applied methods of the criteria scalarization can be altered in order to achieve more complete compliance to available requirements to the optimality. The overcoming of the computational complexity of the developed approach is provided by means of the reuse of the whole search information obtained in the course of computations. The performed numerical experiments have confirmed the reuse of the search information to allow reducing essentially the amount of computations for solving the optimization problems arising successively.

*Keywords:*Decision making, multicriterial optimization, criteria scalarization methods, global optimization with nonlinear constraints, numerical experiment

# **1. Introduction**

The multicriterial optimization (MCO) problems, which are used as the statements of the decision-making problems often are the objects of extensive research - see, for example , the monographs [1-6] and the reviews of the scientific and applied results in the field [7-10].

Usually, the finding of the effective (non-dominated) decisions, in which the improvement of the values with respect to any criteria cannot be achieved without the worsening of the indicators of efficiency with respect to other criteria is understood as the solution of a MCO problem. In the most general case, when solving the MCO problems, it can appear to be necessary to obtain a complete set of the effective decisions (the Pareto set). However, the finding of all effective decisions may require a considerable amount of computations, and the set of obtained decisions may appear to be quite large, the analysis of which may appear to be difficult. As a result,

are applied wider.

the approaches to solving the MCO problems, in which, according to the requirements of optimality, the obtained set of effective decisions is more limited. Among such approaches, there are various kinds of the criteria convolutions, the lexicographic optimization methods, the algorithms of searching the best approximation to the existing to existing prototypes, etc. All methods listed above allow accounting for the specific features of the MCO problems being solved and satisfy the requirements to the optimality from the decision-making person (DMP, decision maker, DM).

The present work is devoted to the solving of the MCO problems, which are used for the description of the complex decision making problems, in which the criteria of efficiency may have a complex *multiextremal* form, and the determining of the values of the criteria and constraints may require *a large amount of computations*. Also, is assumed that in the course of computations it is possible to change the statement, the methods, and the parameters of solving the MCO problem that results in the necessity of the multiple solving of the global optimization problems.

The realism of this approach implies the overcoming of a **considerable computational complexity** of the decision-making problems that can be provided by means of the use of highly efficient global optimization methods and the complete utilization of the search information obtained in the course of computations.

In the present paper, the results of investigations on the generalization of the decision-making problem statements [11-12] and on the development of the highly efficient global optimization methods utilizing the whole search information obtained in the course of computations [13-15] are presented.

Further structure of the paper is as follows. In Section 2, the statement of the decision making problems based on multistage multicriterial global search are presented. In Section 3, a general scheme for the MCO problem criteria of scalarization involving various kinds of the criteria convolutions, the lexicographic optimization methods, and the search for the best approximations of the decisions defined *a priori* is proposed. In Section 4, the search information obtained in the course of computations, and which may be reused in the solving of all subsequent global optimization problems is considered. In Section 5, an efficient algorithm for solving the time-consuming global optimization problems with the nonlinear constraints is presented. Section 6 contains the results of numerical experiments confirming the developed approach to be promising. In Conclusion, the obtained results are discussed and possible main directions of further research are outlined.

# **2. Multiple multicriterial optimization problem statement**

For the formal description of the process of the search for the rational decisions in the complex decision-making problems, the following generalized two-phase model is proposed.

1. In the most general form, a decision-making problem is defined by means of a *vector function of characteristics*

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| Image, Image, | (1) |

where Image is the *vector of design parameters* and Image is the *domain of possible values*, which is usually an *N*-dimensional hypercube

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| Image, | (2) |

for given vectors Image and Image.

It is supposed, that the values of characteristics Image are non-negative, and the decreasing of these ones corresponds to the increasing of the efficiency of the chosen decisions. Also, it is supposed that the characteristics Image may be multiextremal, and the determining of their values may require large enough amount of computations. Besides, the characteristics Image are supposed to satisfy the Lipschitz condition

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| Image, | (3) |

where *Li* is the Lipschitz constant for the characteristic Image and  denotes the Euclidean norm in .

2. Then, a MCO problem is formulated on the basis of the model considered above. For this purpose, a *vector criterion of efficiency* is selected among the characteristics Image from (1)

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| Image | (5) |

and the *vector function of constraints*

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| Image, Image, | (6) |

where Image are the allowances on the feasible values of characteristics Image.

The efficiency criteria and constraints formulated in such a way allow defining a *multicriterial optimization problem*

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| Image, Image, | (7) |

where Imageis the *feasible multicriterial search domain*

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| --- | --- |
| Image. | (8) |

The scheme proposed above involves many existing statements of the optimization problems. At Image and Image, the general statement becomes a global optimization problem. At Image and Image, the general statement defined a nonlinear programming problem. At Image and Image, the general statement leads to a constrained multicriterial optimization problem.

In development of this scheme of the MCO problem statement, further an opportunity of simultaneous formulation of several MCO problems

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| Image | (9) |

will be allowed, the set of which can be varied in the course of computations by means of adding new or by removing already existing problems.

In the simplest case, the set Image can consist of a single MCO problem, upon solving of which current MCO problem statement may be changed after the analysis of the obtained results, and the process of computations can be continued for the new MCO problem statement until the desired optimal decision is obtained. It worth noting that the problems from the set Image are the information compatible ones – when altering the MCO problem statements, the whole search information obtained earlier in the course of computations can be saved and reused in the solving of newly formulated optimization problems.

In general, the proposed model of the optimal decision search process (1) - (10) defines a new class of the optimization problems – the *multiple multicriterial global optimization* (MMGO) problems.

# **3. Reduction of the multiple multicriterial search to the scalar one-dimensional global optimization problems**

One of the approaches to solving the MCO problems used wider consist in the scalarization of the vector criterion into some general scalar criterion of efficiency that allows using a wide set of already existing optimization methods for solving the MCO problems. Among the possible scalarization methods, there are, for example, the weighted sum method, the compromise programming method, the weighted min-max method, and many other methods – see, for example, [2-6].

In the general form, the statement of the global optimization problems generated in the MCO problem criteria scalarization can be represented as follows:

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| Image, Image, | (10) |

where Image is the objective function generated as a result of scalarization of the criteria *fi*, 1*i**s,* Image is the vector of parameters of the applied criteria convolution, Image are the constraints of the MCO problem from (6), and Image is the search domain from (2).

Particular form of the function Image is defined by the criteria scalarization method applied. For example, the following scalarization methods are possible.

1. In the case of equal importance of the criteria *fi*, 1*i**s,* the min-max convolution [4,6]:

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| Image, | (11) |
| Image*s*. |  |

is applied the most often. The necessity and essence of this scalarization method for solving a MCO problem is an important property of the min-max convolution: any result of minimization of Imageleads to obtaining an effective decision and, vise versa, any effective decision can be obtained as a result of minimization of Image at the corresponding values of the convolution coefficients *i*, 1*i**s*.

2. In the case of arrangement of the criteria in importance, the method of successive concessions (MSC) [2,5,6] is used widely. According to this method, the optimization is performed for the most important criterion Image first (the criteria are supposed to be renumbered according to the order of decreasing of their importance). Then, upon completing the global search for Image, the magnitude of allowed concession from the minimum value of the first criterion is set, and the optimization of the second (in the importance) criterion Image at the condition of not exceeding the set concession. Further optimization of the rest criteria is performed in the same way – more detailed description of the MSC method is given, for example, in [2,5,6].

Within the framework of the developed approach, it is proposed to reduce the multistage computations in the MSC method to solving a single scalar optimization problem arising at the last stage of the method of successive concessions

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| Image, Image, | (12) |

Imageare the minimum and maximum values of the criteria in the feasible domain *Q* respectively, and Image are the concessions with respect to each criterion. As before, the values of concessions Image can be varied in the course of computations. The quantities Image the values of which may be unknown *a priori*, can be replaced by the minimum and maximum estimates of the criteria values computed using the available search information.

3. In the case of availability of any estimates of the criteria values of the required decision (for example, based on an ideal decision or on any existing prototype) the MCO problem solution may consist in finding an effective decision the most completely matching given indicators of optimality. Such a problem can be formulated in the form of a scalar optimization problem:

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| Image, Image, | (13) |

where the objective function Image is the standard deviation of the decision Image from the sought ideal decision, and the quantities Image are the magnitudes of importance of approximations with respect to each particular variable Image Image.

The scalarization methods considered above allow accounting for the specific properties of the MCO problems being solved in order to select a desired subset of the effective decisions. Within the framework of the developed approach, it is possible to change the used scalarization methods (11)-(13) and/or altering the parameters of convolutions Image и Image. Such variations expand the set of the MCO problems Image from (9) necessary for solving the original decision making problem into a wider set of the scalar global optimization problems (10)

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| Image | (14) |

in which each problem Image from (9) can correspond to several global optimization problems (10).

In the developed approach, one more step of converting the problems being solved *F*(**,*y*) from (10) is performed, namely the dimensionality reduction is performed with the use of the Peano *space-filling curves* (*evolvents*) *y*(*x*) providing an unambiguous mapping of the interval [0,1]onto an *N*-dimensional hypercube *D* [16, 17]. As a result of such reduction, the multidimensional global optimization problem (10) is reduced to a one-dimensional problem:

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| *F*(**,*y*(*x\**)) *=* min *F*(**,*y*(*x*))  Image*x*[0*,*1]*.* | (15) |

The dimensionality reduction allows applying many well known highly efficient one-dimensional global optimization algorithms for solving the problems (10) (after performing necessary generalization) – see, for example, [16-20].

# **4. Improvement of the efficiency of the multiple multicriterial search on the basis of the reuse of the search information**

The numerical solving of the global optimization problems (10) is usually reduced to the successive computing the values of characteristics Image at the points Image of the search domain Image [16,19]. The data obtained as a result of computations can be represented in the form of the *matrix of the search information*:

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| Image | (16) |

As a result of scalarization of the vector criterion (11) and application of the dimensionality reduction (15), the set Image from (16) can be transformed into the form of the *matrix of the search state*:

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| Image, | (17) |  |

where *xi*,1≤*i*≤*k* are the reduced points of executing the global search iterations arranged in the order of increasing coordinates *zi*, Image, 1≤*i*≤*k* are the values of the scalar criterion and constraints of current optimization problem being solved *F*(**,*y*(*x*)) at these points, and *li*, 1≤*i*≤*k*, are the indices of global search iterations, in which the points *xi*, 1≤*i*≤*k* were computed.

The availability of the search information in the form of the matrices Image and Image allows performing an adaptive choice of the points to perform the global search iterations taking into account the results of all computations completed earlier:

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| Image, | (18) |

(particular form of the rule Image depends on the properties of the optimization algorithm applied).

The availability of the set Image from (16) allows reducing the results of all preceding computations *zi*, 1≤*i*≤*k* in the matrix Image to the values of the next optimization problem being solved *F*(**,*y*(*x*)) from (10) without any repeated time-consuming computations of the values of Image from (1) i. e.

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| Image. | (19) |

This way, all search information can be employed for continuing the computations in full amount. The reuse of the search information can provide a gradual decreasing of the amount of computations when solving every next optimization problem down to the execution of few iterations only to find the next effective decision.

# **5. Efficient method of solving the multiple multicriterial optimization problems with nonlinear constraints**

Within the framework of developed approach, the original method of separate accounting for the constraints developed in the framework of the information –statistical theory of global search [16] was applied for solving the global optimization problems (10) if the nonlinear constraints are available. The essence of the approach is the constructing of a problem with some integral unconstrained objective function, the solving of which leads to the solution of the initial problem (10) – more detailed description of the approach will be given below.

Let us introduced a simpler notation for the reduced one-dimensional problems (10) as

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| min  *gm*+1(*x*)  *gi*(*y*(*x*)) 0*,* 1*i**m, x*[0,1] .  *gm*+1(*x*) = *F(,y*(*x*)). | (20) |

The problem (20) can be considered in the statement of partial computability when each function *gj,*1*j**m+*1 is defined and computable in certain subinterval *j*0,1 only where

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| --- | --- |
| **1** 0,1*, j+*1**  *x**j  gj*(*y*(*x*))0, 1*j**m.* | (21) |

Taking into account the conditions (21), the objective function of the problem (20) can be represented in the form

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| **(*x\**)**min*gm+1*(*y*(*x*))*: x**m+*1*,* | (22) |

on the basis of which, a unified function

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| Image(*y*(*x*))  *g*(*y*(*x*)), ****(*x*), *x*[0,1], | (23) |
| 1**=**(*x*)*m*+1, *g*(*y*(*x*))>0, *gj*(*y*(*x*))0, 1*j***1, |  |

can be constructed. The *index *(*x*), 1***m*+1 defines the first violated constraint at the point *x* in the series.

Determining The value of Image(*y*(*x*))*, x*0,1 is reduced to the successive computing of the quantities *gj*(*y*(*x*))*,* 1*j**(x).* At that, the next value of *gj+*1(*x*) is computed in the case if *gj*(*x*)**0 only. The process of computing is terminated either as a result of funding the inequality *gj*(*x*)*>*0 or as a result of achieving the value **(*x*)*=m+*1 (this procedure is hereafter called a *trial*).

The definition of the function Image(*x*) from (23) allows transforming the problem (20) into an unconstrained optimization problem

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| (*x*\*) ** min(*x*)*: x*[0,1], | (24) |

where

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It is worth noting that the maximum index *M*, the vales of the Lipschitz constant *Image,*1***M*, of the functions *gi*(*y*(*x*)), 1*i**m*+1, and the minimum value  of the function *gm*+1(*y*(*x*)) are unknown. However, when performing the computations, instead of these quantities, the adaptive estimates of these ones can be used, which can be obtained utilizing the search information Image from (17).

Within the framework of the developed approach, the algorithm of global constrained optimization (AGSO), which is considered in details in [16] is applied for solving the problems (24). The general computational scheme of this algorithm can be described briefly as follows.

The first trial is performed at an arbitrary point *x*1(0,1)*.* The choice of any next trial point *xk**, k*1 is determined by the following rules.

*Rule* 1. Renumber the points *x*1*,…, xk* of the preceding trials by the lower indices in the order of increasing coordinate values i. e.

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| *x*1***…****xi*…*xk,* | (25) |

and associate the values *zig*(*xi*)*, *(*xi*)*,* 1*i**k* from (26) with these ones.

*Rule 2*. Evaluate the maximum index *M* and compute the numerical estimates of the Lipschitz constants *Image,*1***M*, of the functions *gi*(*y*(*x*)), 1*i**m*+1, and of the minimum value  of the function *gm*+1(*y*(*x*)) necessary to construct the function (*x*) from (24).

*Rule 3*. For each interval (*xi*1*,xi*)*,* 1<*i**k,* compute the *characteristic R*(*i*) and determine the interval (*xt**,x*t*),* to which the maximum characteristic corresponds

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| *R*(*t*)**max*R*(*i*) 1<*i**k* *.* | (26) |

*Rule 4*. Execute the next trial at the point of the interval *xk*+1(*xt**,xt*).

The iterations of the algorithm are stopped if the stop condition is satisfied

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| *t,* | (37) |

where *t* is from (26) and *0* is the predefined accuracy.

The values of characteristics of the intervals *R*(*i*), 1<*i**k* and the point of the next trials *xk*+1 in the interval with the maximum characteristic are computed according to the rules of the AGCO algorithms. At that, the characteristics of the intervals *R*(*i*)*,*1<*i**k* are defined in such a way, that their values can be interpreted as some measures of importance of the intervals with respect to containing the global minimum point of the function (*x*) from (24).

A detailed description of the AGCO algorithm and the corresponding theory of convergence are presented in [16].

# **6. Results of numerical experiments**